

# LOWER BOUNDS FOR TOPOLOGICAL COMPLEXITY

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**ABSTRACT.** We introduce fibrewise Whitehead- and fibrewise Ganea definitions of topological complexity. We then define several lower bounds for the topological complexity, which improve on the standard lower bound in terms of nilpotency of the cohomology ring. The relationships between these lower bounds are studied.

## 1. INTRODUCTION

The notion of topological complexity was first introduced by Farber in [4]. It is a homotopy invariant that for a given space  $X$  measures the complexity of the problem of determining a path in  $X$  connecting two points in a manner that is continuously dependent on the end-points. In order to give a formal definition observe that the map that assigns to a path  $\alpha: I \rightarrow X$  its end-points  $\alpha(0)$  and  $\alpha(1)$  determines the *path-fibration*  $\text{ev}_{0,1}: X^I \rightarrow X \times X$ . A continuous choice of paths between given end-points corresponds to a continuous section of that fibration.

**Definition 1.** Topological complexity  $\text{TC}(X)$  of a space  $X$  is the least integer  $n$  for which there exist an open cover  $\{U_1, U_2, \dots, U_n\}$  of  $X \times X$  and sections  $s_i: U_i \rightarrow X^I$  of the fibration  $\text{ev}_{0,1}: X^I \rightarrow X \times X$ ,  $\alpha \mapsto (\alpha(0), \alpha(1))$ .

$$\begin{array}{ccc} & & X^I \\ & \nearrow s_i & \downarrow \text{ev}_{0,1} \\ U_i & \xrightarrow{\subseteq} & X \times X \end{array}$$

Topological complexity and its variants are still an intensely studied topic and we do not seem to be close to a definitive formulation. For a sampler of results one can consult Chapter 4 of Farber's book [5].

Clearly,  $\text{TC}(X) = 1$  if and only if  $X$  is contractible. In general the determination of the topological complexity is a non-trivial problem even for simple spaces like the spheres. The standard approach is to find estimates of the topological complexity in the form of upper and lower bounds that are more easily computable. With some luck we then obtain a precise value of  $\text{TC}$  or at least a restricted list of possibilities.

The most common upper bounds are based on the Lusternik-Schnirelmann (LS-)category of the product space  $X \times X$ , and on certain product inequalities. On the other side, the standard lower bound for the topological complexity is the so called zero-divisors cup length  $\text{zcl}(X)$  introduced by Farber in [4] and defined as follows. One takes the cohomology of  $X$  with coefficients in some field  $k$  and defines the

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*zero-divisors cup length*  $\text{zcl}_k(X)$  to be the biggest  $n$  such that there is a non-trivial product of  $n$  elements in the kernel of the cup-product map  $H^*(X; k) \otimes H^*(X; k) \rightarrow H^*(X; k)$ .

We will show that the zero-divisors cup length is only one (and in fact the coarsest) of lower bounds for the topological complexity that can be defined using the characterization of TC by Iwase and Sakai from [9]. Their approach is more geometric - they view the topological complexity as a special case of the *fibrewise pointed Lusternik-Schnirelman category* as defined by James and Morris in [12].

In general, a *fibrewise pointed space* over a *base*  $B$  is a topological space  $E$ , together with a *projection*  $p: E \rightarrow B$  and a *section*  $s: B \rightarrow E$ . Fibrewise pointed spaces over a base  $B$  form a category and the notions of fibrewise pointed maps and fibrewise pointed homotopies are defined in an obvious way. We refer the reader to [11] and [12] for more details on fibrewise constructions.

Iwase and Sakai [9] considered the product  $X \times X$  as a fibrewise pointed space over  $X$  by taking the projection to the first component and the diagonal section  $\Delta: X \rightarrow X \times X$  as in the diagram

$$\begin{array}{c} X \times X \\ \Delta \updownarrow \text{pr}_1 \\ X \end{array}$$

We will denote this fibrewise pointed space by  $X \ltimes X$ . In this case a fibrewise pointed homotopy is any homotopy  $H: X \times X \times I \rightarrow X \times X$  that fixes the first coordinate and is stationary on  $\Delta(X)$  (i.e.  $H(x, y, t) = (x, h(x, y, t))$  for some homotopy  $h: X \times X \times I \rightarrow X$  that is stationary on  $\Delta(X)$ ). The following result of Iwase and Sakai gives us an alternative characterization of the topological complexity:

**Theorem 2** (Iwase-Sakai [9]). *The topological complexity  $\text{TC}(X)$  of  $X$  is equal to the least integer  $n$  for which there exists an open cover  $\{U_1, U_2, \dots, U_n\}$  of  $X \times X$  such that each  $U_i$  is compressible to the diagonal via a fibrewise homotopy.*

*The monoidal topological complexity  $\text{TC}^M(X)$  of  $X$  is equal to the least integer  $n$  for which there exists an open cover  $\{U_1, U_2, \dots, U_n\}$  of  $X \times X$  such that each  $U_i$  contains the diagonal  $\Delta(X)$  and is compressible to the diagonal via a fibrewise pointed homotopy.*

One of the main results of [9] is that  $\text{TC}(X) = \text{TC}^M(X)$  if  $X$  is a locally finite simplicial complex. Recently an error has been discovered in the original proof, and the authors have proposed a new one (cf. [10, Errata]), which proves that  $\text{TC}(X) = \text{TC}^M(X)$  when the minimal cover  $\{U_1, U_2, \dots, U_n\}$  meets certain additional assumptions. These assumptions may be difficult to verify if TC is known but no explicit cover is given, or if the cover provided does not satisfy the conditions. In [6, Section 3] we have been able to improve their argument and show that  $\text{TC}(X) = \text{TC}^M(X)$  when  $X$  is a finite simplicial complex. In view of the homotopy invariance of the topological complexity, the results of this paper hold for spaces that have the homotopy type of a finite CW-complex, which is sufficient for our purposes.

An open set  $U \subseteq X \times X$  is said to be *fibrewise pointed categorical* if it contains the diagonal  $\Delta(X)$  and is compressible onto it by a fibrewise pointed homotopy. We may therefore say that  $\text{TC}(X) \leq n$  if  $X \times X$  can be covered by  $n$  fibrewise pointed categorical sets, a formulation that is completely analogous to the definition of the

LS-category. The principal objective of this paper is to develop further this point of view and to extend some standard constructions from the LS-category to the new context. In particular, we will give a systematic account of lower bounds for the topological complexity that closely follows analogous bounds for the LS-category.

In the first section we will describe fibrewise pointed spaces that will play a role in the later exposition. The notation is chosen so as to stress the relation with the fibrewise pointed space  $X \ltimes X$  used in the Iwase-Sakai definition. The common framework that justifies the same notation for seemingly disparate constructions is given in the Appendix.

In the second section we will describe the Whitehead- and the Ganea-type formulations of the topological complexity. Both formulations are of course known: the former appears in [9] and can be in fact traced back at least to [12, Section 6]; the latter is easily seen to be equivalent to the Schwarz genus [13] of the path-fibration, which as a characterization of the topological complexity appears in the original paper [4] by Farber. We need both formulations in order to construct a diagram of fibrewise pointed spaces needed to describe lower bounds for the topological complexity.

In the last section we describe a series of estimates for the topological complexity and derive principal relations between them. Variants of some of these lower bounds have already been considered in [9, Section 8] and [7].

Throughout this paper  $1$  is used to denote the identity map when the domain can be inferred from the context,  $\Delta_n : A \rightarrow A^n$  denotes the diagonal map  $a \mapsto (a, \dots, a)$  (we write  $\Delta$  instead of  $\Delta_2$  to unburden the already encumbered notation) and  $\text{pr}_1 : A \times B \rightarrow A$  denotes the projection to the first component for any spaces  $A, B$ . The symbol  $\mathbb{N}$  will denote the set of positive integers and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ .

## 2. POINTED CONSTRUCTIONS AND FIBREWISE POINTED SPACES

Several lower bounds for the LS-category can be derived from a diagram that relates the Whitehead and the Ganea characterization (see Section 1.6 and Chapter 2 of [1]). To obtain analogous estimates for the topological complexity, we must first understand the fine structure of the fibrewise pointed spaces involved. In all cases we obtain fibrewise pointed spaces that can be seen as continuous families of pointed topological spaces whose base-points are parametrized by the points of the base space.

We are going to describe a number of fibrewise pointed spaces that play a role in the fibrewise Whitehead and fibrewise Ganea definitions of TC. These are of course only special instances of general fibrewise pointed constructions described in [2, Chapter 2], but in our case they allow a very explicit description that will play a role in our exposition. In spite of the different definitions these spaces share some basic features and this motivates the common notation. We will provide further justification for this decision in the Appendix, where we will describe a general construction, which under favourable circumstances subsumes all of the above. In the following examples let  $X$  be any topological space.

**Pointed space:** Let  $X \ltimes X$  denote the fibrewise pointed space  $X \xrightarrow{\Delta} X \times X \xrightarrow{\text{pr}_1} X$ . As a space  $X \ltimes X$  is therefore just the product  $X \times X$  but the choice of the section means that the fibre over each  $x \in X$  is the pointed topological space  $(X, x)$  rather than  $(X, x_0)$  for some fixed  $x_0 \in X$ .

**Product:** The categorical product in the category of fibrewise spaces is given by the pull-back construction over the base space. If we compute the fibrewise pointed  $n$ -fold product of  $X \ltimes X$  we obtain the space  $\{(x_1, y_1, \dots, x_n, y_n) \in X^{2n} \mid x_1 = \dots = x_n\}$ , with the projection  $pr_1: (x_1, y_1, \dots, x_n, y_n) \mapsto x_1$  and the diagonal section  $\Delta_{2n}$ . Since the odd-numbered coordinates in the product coincide we may instead take the space  $\{(x, y_1, \dots, y_n) \in X \times X^n\}$ , which is a fibrewise pointed space with respect to the projection  $pr_1$  and the section  $(1, \Delta_n): X \rightarrow X \times X^n$ . We denote this *fibrewise pointed  $n$ -fold product* by  $X \ltimes \Pi^n X$ . Observe that the fibre over  $x \in X$  is the pointed product space  $(X^n, (x, \dots, x))$ .

**Fat wedge:** From the  $n$ -fold product we may extract the *fibrewise pointed fat wedge* by taking the subspace  $\{(x, y_1, \dots, y_n) \in X \ltimes \Pi^n X \mid \exists j : y_j = x\}$  and restricting the projection and the section accordingly. This fibrewise pointed space is denoted  $X \ltimes W^n X$ . The fibre over  $x \in X$  is the usual fat wedge  $W^n(X, x) = \{(x_1, \dots, x_n) \in X^n \mid \exists j : x_j = x\}$ . Observe that, contrary to the first two examples, the fibres of  $X \ltimes W^n X$  are in general not homeomorphic. This shows that  $X \ltimes W^n X$  may not be locally trivial. There is an obvious fibrewise pointed inclusion  $X \ltimes W^n X \hookrightarrow X \ltimes \Pi^n X$ , denoted  $1 \ltimes i_n$ .

**Smash product:** The  $n$ -fold *fibrewise pointed smash product*  $X \ltimes \wedge^n X$  is obtained by taking the fibrewise quotient of  $X \ltimes \Pi^n X$  by its subspace  $X \ltimes W^n X$ . The projection  $pr: X \ltimes \wedge^n X \rightarrow X$  and the section  $s: X \rightarrow X \ltimes \wedge^n X$  are induced by the projection and the section in the  $n$ -fold product. The quotient map  $X \ltimes \Pi^n X \rightarrow X \ltimes \wedge^n X$  is denoted  $1 \ltimes q_n$ , and the sequence of fibrewise maps and spaces

$$X \ltimes W^n X \xrightarrow{1 \ltimes i_n} X \ltimes \Pi^n X \xrightarrow{1 \ltimes q_n} X \ltimes \wedge^n X$$

is a fibrewise cofibration in the sense of [2].

**Path space:** Let  $ev_0: X^I \rightarrow X$  be the projection that to every path in  $X$  assigns its starting point, and let  $c: X \rightarrow X^I$  be the map that to every  $x \in X$  assigns the stationary path at the point  $x$ . The fibre of  $ev_0$  over any  $x \in X$  is precisely  $P_x X$ , the space of paths in  $X$ , based at the point  $x$ . It is therefore natural to denote by  $X \ltimes PX$  the fibrewise pointed space  $X \xrightarrow{c} X^I \xrightarrow{ev_0} X$ . Observe that the fibration  $ev_{0,1}: X^I \rightarrow X \times X$  determines the fibrewise pointed fibration  $X \ltimes PX \rightarrow X \ltimes X$ .

**Ganea space:** By analogy with the above examples one would expect the fibrewise Ganea space  $X \ltimes G_n(X)$  to be a continuous family of Ganea spaces  $G_n(X)$  (see [1, Section 1.6]) with the base-points varying according to a chosen point in  $X$ . Recall that the standard Ganea construction is obtained by starting from a fibration  $p: E \rightarrow B$  with fibre  $F$  and turning the induced map  $p: E/F \rightarrow B$  into a fibration. Then the path fibration  $ev_1: PX \rightarrow X$  is defined to be the first Ganea fibration  $p_1: G_1(X) \rightarrow X$  for the space  $X$ , and the  $(n+1)$ st Ganea fibration  $p_{n+1}: G_{n+1} \rightarrow X$  is obtained recursively by applying the Ganea construction to the fibration  $p_n$ .

However, for practical reasons we prefer the alternative description of the Ganea space based on the fibrewise join operation introduced by Schwarz [13]. Thus we start with the path fibration  $ev_1: PX \rightarrow X$  as before and define the  $n$ th Ganea fibration to be the  $n$ -fold fibrewise reduced join  $PX *_X \dots *_X PX \rightarrow X$ .

The last formulation has a natural fibrewise extension: we define  $X \ltimes G_n(X)$  by taking the  $n$ -fold fibrewise reduced join of the path fibration  $\text{ev}_{0,1}: X^I \rightarrow X \times X$  and composing the projection  $X^I *_{X \times X} \dots *_{X \times X} X^I \rightarrow X \times X$  with  $\text{pr}_1: X \times X \rightarrow X$ . This fibrewise space is pointed with respect to the section  $\widehat{G}_n: X \rightarrow G_n(X)$  that assigns to each  $x \in X$  the point in the total space determined by the stationary path in  $x$ . It is clear that the fibers over points in  $X$  are precisely the previously described Ganea spaces, and that justifies the notation  $X \ltimes G_n(X)$ .

### 3. WHITEHEAD- AND GANEA-TYPE DEFINITIONS

In this section we take as a starting point the Iwase-Sakai approach to obtain formulations of the topological complexity that are completely analogous to the classical Whitehead and Ganea characterizations of the LS-category. This was partly done in [9] but we are able to take a step further and use the fibrewise pointed spaces described in the preceding section to obtain the fibrewise pointed version of the standard diagram that relates the two characterizations (cf. [1, Section 1.4]). That diagram will be used in Section 4 to give a unified treatment and comparison of various estimates of the topological complexity.

We begin with the Whitehead-type formulation. As usual we need some mild topological assumption to prove the equivalence with the original definition. Recall that the space  $X$  is *fully normal* if  $X \times I^n$  is normal for all  $n \in \mathbb{N}_0$ .

**Theorem 3** (Whitehead-type definition of TC). *Let  $X \times X$  be a fully normal Hausdorff space. Then  $\text{TC}(X) \leq n$  if and only if the map  $1 \ltimes \Delta_n: X \ltimes X \rightarrow X \ltimes \Pi^n X$  can be compressed into  $X \ltimes W^n X$  by a fibrewise pointed homotopy.*

$$\begin{array}{ccc} & & X \ltimes W^n X \\ & \nearrow g & \downarrow 1 \ltimes i_n \\ X \ltimes X & \xrightarrow{1 \ltimes \Delta_n} & X \ltimes \Pi^n X \end{array}$$

*Proof.* This theorem can be viewed as a special case of Propositions 6.1 and 6.2 of [12]. We summarize their proof but adjust for this particular setting.

First, let  $H_t: X \ltimes X \rightarrow X \ltimes \Pi^n X$  be the fibrewise pointed deformation of  $1 \ltimes \Delta_n$  into a map  $g: X \ltimes X \rightarrow X \ltimes W^n X$ . Let  $U$  be an open fibrewise pointed categorical neighbourhood of  $\Delta(X)$ . Define  $U_i = g^{-1}(1 \ltimes \text{pr}_i)^{-1}(U)$  where  $1 \ltimes \text{pr}_i: X \ltimes \Pi^n X \rightarrow X \ltimes X$  is  $i$ th projection on the second factor. Then

$$(1 \ltimes \text{pr}_i)H_t: X \ltimes X \rightarrow X \ltimes X$$

compresses  $U_i$  to  $U$  via a fibrewise pointed homotopy. So,  $\{U_1, \dots, U_n\}$  is a fibrewise pointed open categorical cover of  $X \ltimes X$  and  $\text{TC}(X) \leq n$ .

Conversely, assume that  $\text{TC}(X) \leq n$  and let  $\{V_1, \dots, V_n\}$  be the fibrewise pointed open categorical cover of  $X \ltimes X$ . Let  $H_i: V_i \rightarrow X \ltimes X$  be the fibrewise pointed nullhomotopy of the inclusions  $j_i: V_i \rightarrow X \ltimes X$ ,  $i = 1, \dots, n$ . Since  $X \ltimes X$  is normal and  $\Delta(X)$  is closed in  $X \ltimes X$  there exist open neighbourhoods  $W_i$  of  $\Delta(X)$  such that  $\overline{W}_i \subset V_i$  and maps  $r_i: X \ltimes X \rightarrow I$  with  $r_i|_{\Delta(X)} = 1$  and  $r_i|_{X \ltimes X \setminus W_i} = 0$ . A fibrewise pointed deformation  $d_i: (X \ltimes X) \times I \rightarrow X \ltimes X$  of the identity is given by

$$d_i(x, y, t) = \begin{cases} (x, y); & (x, y) \in X \ltimes X \setminus \overline{W}_i, \\ H_i(x, y, t \cdot r_i(x, y)); & (x, y) \in V_i. \end{cases}$$

Let  $d: (X \times X) \times I \rightarrow X \times \Pi^n X$  be the fibrewise pointed homotopy defined by the relations  $(1 \times \text{pr}_i)d = d_i$  for  $i = 1, \dots, n$ . Then  $d$  compresses  $1 \times \Delta_n$  into  $X \times W^n X$ .  $\square$

The Ganea definition of the LS-category is based on the homotopy pull-back of the diagram used in the Whitehead definition. In fact, if we let  $\overline{G}_n(X) = \{\alpha \in (\Pi^n X)^I \mid \alpha(0) \in W^n X, \alpha(1) \in \Delta_n(X)\}$  then the diagram

$$\begin{array}{ccc} \overline{G}_n(X) & \longrightarrow & W^n X \\ \downarrow & & \downarrow i_n \\ X & \xrightarrow{\Delta_n} & \Pi^n X \end{array}$$

(with obvious projections from  $\overline{G}_n(X)$ ) is the standard homotopy pull-back. The space  $\overline{G}_n(X)$  is known to be homotopy equivalent to the previously defined Ganea space  $G_n(X)$  (cf. [1, Theorem 1.63]). It is therefore reasonable to consider the fibrewise homotopy pull-back of the diagram used in the Whitehead definition for an alternative formulation of the topological complexity.

We begin by replacing the inclusion  $1 \times i_n: X \times W^n X \rightarrow X \times \Pi^n X$  by the fibration

$$\begin{aligned} 1 \times \overline{t}_n: X \times (W^n X \sqcap (\Pi^n X)^I) &\rightarrow X \times \Pi^n X, \\ (1 \times \overline{t}_n)(x, y_1, \dots, y_n, \alpha) &= (x, \alpha(1)), \end{aligned}$$

where the domain space is given by

$$E := \{(x, y_1, \dots, y_n, \alpha) \in X \times W^n X \times (\Pi^n X)^I \mid \exists j: y_j = x, \alpha(0) = (y_1, \dots, y_n)\}.$$

The fibrewise pullback along  $1 \times \Delta_n$  gives us the space

$$\overline{E} := \{(x', y', x, y_1, \dots, y_n, \alpha) \in X \times X \times E \mid (y, y, \dots, y) = \alpha(1), x' = x\}.$$

that can be also written in a more compact form as

$$\overline{E} = \{(x, \alpha) \in X \times (\Pi^n X)^I \mid (x, \alpha(0)) \in X \times W^n X, \alpha(1) \in \Delta_n(X)\}.$$

Let  $p: \overline{E} \rightarrow X$  be the projection to the first component, and let  $s: X \rightarrow \overline{E}$  be given by  $x \mapsto (x, c_x, \dots, c_x)$ . Since the fibre of  $p$  over any  $x \in X$  is precisely the space  $\overline{G}_n(X)$  based at the point  $(c_x, \dots, c_x) = \Delta_n(c_x)$ , we denote the fibrewise pointed space  $X \xrightarrow{s} \overline{E} \xrightarrow{p} X$  by  $X \times \overline{G}_n(X)$ . The fibrewise pointed spaces  $X \times G_n(X)$  and  $X \times \overline{G}_n(X)$  are related by a fibrewise map which is a homotopy equivalence when restricted to each fibre. By Theorem 3.6 of [3] they are fibre-homotopy equivalent. Thus we may conclude that the diagram

$$(1) \quad \begin{array}{ccc} X \times G_n X & \longrightarrow & X \times W^n X \\ 1 \times p_n \downarrow & & \downarrow 1 \times i_n \\ X \times X & \xrightarrow{1 \times \Delta_n} & X \times \Pi^n X \end{array}$$

is a fibrewise pointed homotopy pull-back. Since the liftings of  $1 \times \Delta_n$  correspond to sections of  $1 \times p_n$  we have the following

**Corollary 4** (Ganea-type definition of TC). *TC(X) is the least integer n, such that the map  $1 \times p_n: X \times G_n(X) \rightarrow X \times X$  admits a section.*

**Remark 5.** Observe that we have just reobtained Farber's original definition of the topological complexity. Indeed, Farber defined  $\text{TC}(X)$  to be the Schwarz genus of the path-fibration  $\text{ev}_{0,1}: X^I \rightarrow X \times X$ , and Schwarz proved in [13] that the genus of a fibration  $p: E \rightarrow B$  is the minimal  $n$  such that the  $n$ -fold fibrewise join  $p_n: E *_B \dots *_B E \rightarrow B$  admits a section. Since in Section 2 we defined  $X \ltimes G_n(X)$  as the  $n$ -fold fibrewise join of the path-fibration, we see that Farber's definition coincides with the Ganea-type definition of the fibrewise pointed LS-category.

The fibrewise Ganea- and Whitehead-type definitions of topological complexity are analogous to the Ganea and Whitehead definitions of LS-category. Indeed, if we choose any  $x \in X$  and restrict all spaces and maps in the diagram above to the fibre over  $x$ , we retrieve the homotopy pullback diagram for the LS-category. This leads to the following diagram of fibrewise pointed spaces over  $X$

$$\begin{array}{ccc}
 & G_n(X) & \xrightarrow{\quad} W^n X \\
 & \swarrow \downarrow & \swarrow \downarrow \\
 X & \xrightarrow{\quad} \Pi^n X & \\
 \downarrow & \downarrow & \downarrow \\
 X \ltimes G_n(X) & \xrightarrow{\quad} X \ltimes W^n X & \\
 \downarrow & \downarrow & \downarrow \\
 X \ltimes X & \xrightarrow{\quad} X \ltimes \Pi^n X & \\
 \downarrow & \downarrow & \downarrow \\
 X & \xlongequal{\quad} X & \\
 \downarrow & \downarrow & \downarrow \\
 X & \xlongequal{\quad} X &
 \end{array}$$

that allows us to easily compare any invariants that are connected to the existence of sections in this diagram. For example, we see that  $\text{cat}(X) \leq \text{TC}(X)$  since  $p_n$  obviously admits a section if  $1 \ltimes p_n$  admits a section.

#### 4. LOWER BOUNDS FOR TOPOLOGICAL COMPLEXITY

We have already mentioned the most widely used lower bound for the topological complexity, namely the zero-divisors cup length. Although the original definition used cohomology with field coefficients this restriction is not necessary. Indeed let  $R$  be any ring and let  $\text{nil}_R(X)$  be the least  $n$  such that all cup-products of length  $n$  in the ring  $H^*(X \times X, \Delta(X); R)$  are trivial. Then we have the following estimate (cf. also [12, Section 3]):

**Proposition 6.** *Let  $R$  be a ring and let  $X$  be a locally finite simplicial complex. Then  $\text{TC}(X) \geq \text{nil}_R(X)$ .*

*Proof.* Recall that  $\text{TC}(X)$  is the fibrewise pointed category of the fibrewise pointed space  $X \xrightarrow{\Delta} X \ltimes X \xrightarrow{\text{pr}_1} X$ . If the subset  $U$  is fibrewise pointed categorical in  $X \ltimes X$ , then the induced map

$$H^*(X \ltimes X, \Delta(X)) \longrightarrow H^*(U, \Delta(X))$$

is trivial. It then follows from the cohomology exact sequence

$$\dots \rightarrow H^*(X \ltimes X, U) \rightarrow H^*(X \ltimes X, \Delta(X)) \xrightarrow{0} H^*(U, \Delta(X)) \rightarrow H^*(X \ltimes X, U) \rightarrow \dots$$

of the triple  $(X \ltimes X, U, \Delta(X))$  that the map

$$H^*(X \ltimes X, U) \longrightarrow H^*(X \ltimes X, \Delta(X))$$

is surjective. If  $\{U_1, \dots, U_n\}$  is a fibrewise pointed categorical open covering of  $X \ltimes X$ , then for every  $\alpha_i \in H^*(X \ltimes X, \Delta(X))$  there exists a  $\beta_i \in H^*(X \ltimes X, U_i)$ ,  $i = 1, \dots, n$ . The product

$$\beta_1 \smile \dots \smile \beta_n \in H^*(X \ltimes X, U_1 \cup \dots \cup U_n)$$

is zero since the group itself is trivial. So, the product

$$\alpha_1 \smile \dots \smile \alpha_n \in H^*(X \ltimes X, \Delta(X))$$

is zero. This shows that  $\text{TC}(X) \geq \text{nil}_R(X)$ .  $\square$

**Remark 7.** If we specialize to coefficients in a field  $k$  then  $\text{nil}_k(X) = \text{zcl}_k(X) + 1$ . In fact, the cohomology exact sequence of the pair  $(X \times X, \Delta(X))$  splits into short exact sequences of the form

$$0 \rightarrow H^*(X \times X, \Delta(X); k) \rightarrow H^*(X \times X; k) \xrightarrow{i^*} H^*(\Delta(X); k) \rightarrow 0$$

It is well-known that the homomorphism  $i^*$  induces the cup-product map from  $H^*(X \times X; k) = H^*(X; k) \otimes H^*(X; k)$  to  $H^*(\Delta(X); k) = H^*(X; k)$ , therefore  $\ker i^* = H^*(X \times X, \Delta(X); k)$  may be identified with the ideal of zero-divisors used in the definition of  $\text{zcl}_k(X)$ .

In the theory of LS-category there are several invariants that are better estimates of  $\text{cat}(X)$  than the cup-length. A systematic exposition of lower bounds for the LS-category in [1, Chapter 2] is based on the diagram relating the Whitehead and Ganea characterizations of the LS-category. Based on the results from the previous section we are now able to generalize this approach and obtain estimates of the topological complexity that are better than the zero-divisors cup length. We first complete the pullback diagram (1) by adding the cofibres of the vertical maps. We have already considered the cofibre of the inclusion  $1 \ltimes i_n: X \ltimes W^n X \rightarrow X \ltimes \Pi^n X$  and we denoted it by  $X \ltimes \wedge^n X$ . We may likewise construct the fibrewise pointed space  $X \ltimes G_{[n]}(X)$  as the cofibre of the map  $1 \ltimes p_n: X \ltimes G_n(X) \rightarrow X \ltimes X$ . The quotient map is denoted  $1 \ltimes q'_n: X \ltimes X \rightarrow X \ltimes G_{[n]}(X)$ . We obtain the following diagram of fibrewise pointed spaces over  $X$ :

$$\begin{array}{ccc} X \ltimes G_n X & \xrightarrow{1 \ltimes \hat{\Delta}_n} & X \ltimes W^n X \\ 1 \ltimes p_n \downarrow & & \downarrow 1 \ltimes i_n \\ X \ltimes X & \xrightarrow{1 \ltimes \Delta_n} & X \ltimes \Pi^n X \\ 1 \ltimes q'_n \downarrow & & \downarrow 1 \ltimes q_n \\ X \ltimes G_{[n]} X & \xrightarrow{1 \ltimes \tilde{\Delta}_n} & X \ltimes \wedge^n X \end{array}$$

Based on the interrelations in the diagram we define several lower bounds for the topological complexity. All of them have analogues among lower bounds for the LS-category and that motivates both the notation and their names.

**Definition 8** (Lower bounds for TC).

- (1) Let  $R$  be a ring. The TC-Toomer invariant of  $X$  with coefficients in  $R$ , denoted  $e\text{TC}_R(X)$ , is the least integer  $n$  such that the induced map

$$H_*(1 \ltimes p_n): H_*(X \ltimes G_n(X), \hat{G}_n(X); R) \rightarrow H_*(X \ltimes X, \Delta(X); R)$$

is surjective.



- (2) The weak topological complexity,  $\text{wTC}(X)$ , is the least integer  $n$ , such that the composition  $(1 \times q_n)(1 \times \Delta_n)$  is fibrewise pointed nullhomotopic.
- (3) The TC-conilpotency of  $X$ ,  $\text{cTC}(X)$ , is the least integer  $n$ , such that the composition  $(1 \times \Sigma q_n)(1 \times \Sigma \Delta_n)$  is fibrewise pointed nullhomotopic.
- (4) For any integer  $j \geq 0$  let  $\sigma^j \text{TC}(X)$  be the least integer  $n$  for which there exists a fibrewise map  $1 \times s: X \times \Sigma^j X \rightarrow X \times \Sigma^j G_n(X)$ , such that the composition  $(1 \times \Sigma^j p_n)(1 \times s)$  is fibrewise pointed homotopic to the identity. In particular,  $\sigma^0 \text{TC}(X) = \text{TC}(X)$ . The  $\sigma$ -topological complexity of  $X$  is

$$\sigma \text{TC}(X) = \inf_{j \in \mathbb{N}} \sigma^j \text{TC}(X).$$

- (5) The weak topological complexity of Ganea  $\text{wTC}_G(X)$  is the least integer  $n$ , such that  $1 \times q'_n$  is fibrewise pointed nullhomotopic.

Table 1 relates these new lower bounds to the corresponding notions for the LS category.

LS CATEGORY	TOPOLOGICAL COMPLEXITY
$e_R(X)$ (Toomer invariant)	$e\text{TC}_R(X)$
$\text{wcat}(X)$ (weak category)	$\text{wTC}(X)$
$\text{conil}(X)$ (conilpotency)	$\text{cTC}(X)$
$\sigma\text{cat}(X)$ ( $\sigma$ -category)	$\sigma\text{TC}(X)$
$\text{wcat}_G(X)$ (weak category of Ganea)	$\text{wTC}_G(X)$

TABLE 1. Lower bounds for TC and their LS category counterparts.

**Remark 9.** A fibrewise pointed map  $f: E_1 \rightarrow E_2$  between the fibrewise pointed spaces  $B \xrightarrow{s_i} E_i \xrightarrow{p_i} B$  is fibrewise pointed nullhomotopic, denoted  $f \simeq_F *$ , if it is fibrewise pointed homotopic to  $s_2 \circ p_1$ .

Let  $X \xrightarrow{s} X \times Y \xrightarrow{p} X$  be any of the fibrewise pointed constructions described in Section 2. We will denote by  $X \times \Sigma Y$  the fibrewise pointed suspension of  $X \times Y$ . To compare the lower bounds we just introduced we need the following result:

**Proposition 10.** For the homology and cohomology with any coefficients we have the following isomorphisms

$$H_n(X \times Y, s(X)) \cong H_{n+j}(X \times \Sigma^j Y, (\Sigma^j s)(X))$$

and

$$H^n(X \times Y, s(X)) \cong H^{n+j}(X \times \Sigma^j Y, (\Sigma^j s)(X)).$$

Moreover, if  $1 \times f: X \times Y \rightarrow X \times Z$  is a fibrewise pointed map then there is a commutative diagram

$$\begin{array}{ccc} H_n(X \times Y, s(X)) & \xrightarrow{H_n(1 \times f)} & H_n(X \times Z, s(X)) \\ \cong \downarrow & & \cong \downarrow \\ H_{n+j}(X \times \Sigma^j Y, (\Sigma^j s)(X)) & \xrightarrow{H_{n+j}(1 \times \Sigma^j f)} & H_{n+j}(X \times \Sigma^j Z, (\Sigma^j s)(X)) \end{array}$$

and a similar result holds for cohomology.

*Proof.* Recall that  $(X \ltimes \Sigma Y, (\Sigma s)(X))$  is obtained by gluing of the fibrewise pointed cones  $(X \ltimes C^+Y, (C^+s)(X))$  and  $(X \ltimes C^-Y, (C^-s)(X))$ . The two cones are fibrewise pointed contractible, so  $H_n(X \ltimes C^+Y, (C^+s)(X)) \cong H_n(X \ltimes C^-Y, (C^-s)(X)) = 0$ . The intersection of the two cones is  $(X \ltimes Y, s(X))$ , so from the relative version of the Mayer-Vietoris sequence for homology we can conclude that

$$H_n(X \ltimes Y, s(X)) \cong H_{n+1}(X \ltimes \Sigma Y, (\Sigma s)(X)).$$

The analogous argument for cohomology deals with the second statement.  $\square$

As a preparation for the next theorem we list the relations between the lower bounds that are clear from the definition.

**Proposition 11.**

- (1)  $\mathrm{TC}(X) \geq e\mathrm{TC}_R(X)$ ,
- (2)  $\mathrm{TC}(X) \geq \mathrm{wTC}(X) \geq c\mathrm{TC}(X)$ ,
- (3)  $\mathrm{TC}(X) \geq \mathrm{wTC}_G(X) \geq \sigma^i\mathrm{TC}(X) \geq \sigma^j\mathrm{TC}(X) \geq \sigma\mathrm{TC}(X)$  for  $1 \leq i \leq j$ ,
- (4)  $\mathrm{wTC}_G(X) \geq \mathrm{wTC}(X)$ .

*Proof.*

- (1) If  $1 \ltimes p_n: X \ltimes G_n(X) \rightarrow X \ltimes X$  has a section then  $H_*(1 \ltimes p_n; k)$  are surjective, so  $\mathrm{TC}(X) \geq e_k^{\mathrm{TC}}(X)$ .
- (2) If there exists a map  $s: X \ltimes X \rightarrow X \ltimes W^n X$  such that  $(1 \ltimes i_n)s \simeq_F 1 \ltimes \Delta_n$ , then  $(1 \ltimes q_n)(1 \ltimes \Delta_n) \simeq_F (1 \ltimes q_n)(1 \ltimes i_n)s \simeq_F *$ , so  $\mathrm{TC}(X) \geq \mathrm{wTC}(X)$ . If  $(1 \ltimes q_n)(1 \ltimes \Delta_n) \simeq_F *$ , then  $(1 \ltimes \Sigma q_n)(1 \ltimes \Sigma \Delta_n) \simeq_F *$ , so  $\mathrm{wTC}(X) \geq c\mathrm{TC}(X)$ .
- (3) Observe that  $\mathrm{wTC}_G(X) = \sigma^1\mathrm{TC}(X)$ . Indeed, it follows from the fibrewise Barratt-Puppe sequence

$$X \ltimes G_n(X) \xrightarrow{1 \ltimes p_n} X \ltimes X \xrightarrow{1 \ltimes q'_n} X \ltimes G_{[n]}(X) \xrightarrow{1 \ltimes \delta}$$

$$\rightarrow X \ltimes \Sigma G_n(X) \xrightarrow{1 \ltimes \Sigma p_n} X \ltimes \Sigma X \xrightarrow{1 \ltimes \Sigma q'_n} X \ltimes \Sigma G_{[n]}(X) \rightarrow \dots$$

that the map  $1 \ltimes q'_n$  is fibrewise pointed nullhomotopic if and only if the map  $1 \ltimes \Sigma p_n$  admits a homotopy section. Let  $1 \leq i \leq j$ . If  $1 \ltimes G_n(X) \rightarrow X \ltimes X$  has a section, then there exists a map  $1 \ltimes s: X \ltimes X \rightarrow X \ltimes G_n(X)$ , such that  $(1 \ltimes p_n)(1 \ltimes s) \simeq_F 1$ , and so  $(1 \ltimes \Sigma^i p_n)(1 \ltimes \Sigma^i s) \simeq_F 1$ . So,  $\mathrm{TC}(X) \geq \sigma^i\mathrm{TC}(X)$ . If  $(1 \ltimes \Sigma^i p_n)(1 \ltimes \Sigma^i s) \simeq_F 1$ , then  $(1 \ltimes \Sigma^j p_n)(1 \ltimes \Sigma^j s) \simeq_F 1$ , so  $\sigma^i\mathrm{TC}(X) \geq \sigma^j\mathrm{TC}(X)$ . It is now obvious that  $\mathrm{TC}(X) \geq \mathrm{wTC}_G(X) \geq \sigma^i\mathrm{TC}(X) \geq \sigma^j\mathrm{TC}(X)$ . The rightmost inequality follows immediately from the definition of  $\sigma\mathrm{TC}(X)$ .

- (4) If  $(1 \ltimes q'_n) \simeq_F *$ , then  $(1 \ltimes q_n)(1 \ltimes \Delta_n) \simeq_F (1 \ltimes \tilde{\Delta}_n)(1 \ltimes q'_n) \simeq_F *$ , so  $\mathrm{wTC}_G(X) \geq \mathrm{wTC}(X)$ .

$\square$

The following theorem summarizes all known relations between the various lower bounds considered in this paper.

**Theorem 12.** *Let  $R$  be a ring. Then*

$$\mathrm{nil}_R(X) \leq \sigma\mathrm{TC}(X) \leq \mathrm{wTC}_G(X) \leq \mathrm{TC}(X) \text{ and}$$

$$\mathrm{nil}_R(X) \leq c\mathrm{TC}(X) \leq \mathrm{wTC}(X) \leq \mathrm{wTC}_G(X) \leq \mathrm{TC}(X).$$

If  $k$  is a field, then

$$\text{nil}_k(X) \leq e_k^{\text{TC}}(X) \leq \sigma\text{TC}(X).$$

*Proof.* If we compare the inequalities above with those from Proposition 11, we see that we only have to show the following four: (a)  $\text{nil}_R(X) \leq \sigma\text{TC}(X)$ , (b)  $\text{nil}_R(X) \leq c\text{TC}(X)$ , (c)  $\text{nil}_k(X) \leq e_k^{\text{TC}}(X)$  and (d)  $e_k^{\text{TC}}(X) \leq \sigma\text{TC}(X)$ .

- (a) Recall that  $\sigma\text{TC}(X) = \inf_{j \in \mathbb{N}} \sigma^j\text{TC}(X)$ . It therefore suffices to show that  $\text{nil}_R(X) \leq \sigma^j\text{TC}(X)$  for all  $j \in \mathbb{N}$  and this will imply that  $\text{nil}_R(X) \leq \sigma\text{TC}(X)$ . Fix  $j$ . If  $\sigma^j\text{TC}(X) = \infty$ , then the statement obviously holds. So, let  $\sigma^j\text{TC}(X) = n < \infty$ . Then there exists a map  $1 \times s: X \times \Sigma^j X \rightarrow X \times \Sigma^j G_n(X)$ , such that  $(1 \times \Sigma^j p_n)(1 \times s) \simeq_F 1$ . We can conclude that the induced maps  $H^*(1 \times \Sigma^j p_n)$  are injective. By Proposition 10 the maps  $H^*(1 \times p_n)$  are injective, so the maps  $H^*(1 \times q'_n)$  are trivial. We conclude that  $H^*(1 \times q_n)H^*(1 \times \Delta_n) = 0$ . Observe that the composition  $(1 \times \Delta_n)(1 \times q_n)$  induces on cohomology precisely the  $n$ -fold cup-product on elements of positive dimension. So,  $\text{nil}_R(X) \leq n$ .
- (b) Let  $c\text{TC}(X) = n$ . Then  $(1 \times \Sigma q_n)(1 \times \Sigma \Delta_n) \simeq_F *$ , so  $0 = H^*((1 \times \Sigma q_n)(1 \times \Sigma \Delta_n)) = H^*((1 \times q_n)(1 \times \Delta_n))$ . Again by definition  $\text{nil}_R(X) \leq n$ .
- (c) Assume that  $e_k^{\text{TC}}(X) = n$  and the maps  $H_*(1 \times p_n)$  are surjective. Then  $H_*(1 \times q'_n) = 0$  and  $k$  is a field, so  $H^*(1 \times q'_n) = 0$ . We conclude that  $H^*(1 \times q_n)H^*(1 \times \Delta_n) = 0$  and  $\text{nil}_k(X) \leq n$ .
- (d) The idea in this case is the same as in (a). Let  $\sigma^j\text{TC}(X) = n < \infty$ . Then there exists a map  $1 \times s: X \times \Sigma^j X \rightarrow X \times \Sigma^j G_n(X)$ , such that  $(1 \times \Sigma^j p_n)(1 \times s) \simeq_F 1$ . Hence,  $H_*(1 \times \Sigma^j p_n)$  are surjective. By Proposition 10 the maps  $H_*(1 \times p_n)$  are surjective and we have  $e_k^{\text{TC}}(X) \leq n$ .

□

## 5. APPENDIX

In Section 2 we used a common notation for a number of different constructions of fibrewise pointed spaces. In all cases the fibres were given by some functorial construction applied to a fixed space but with a varying choice of base-points. The underlying intuition is that the base space acts on the fibres by 'moving around' the base-points and that the resulting fibrewise space is some kind of a semi-direct product of the base with the fibre determined by this action. In this Appendix we are going to put this intuition on a firm basis by describing a general construction which is essentially a fibrewise pointed version of the familiar change-of-fibre procedure. To achieve this we restrict our attention to open manifolds. This is not a real limitation as the construction can be extended by standard techniques to ENRs.

Recall that a fibrewise pointed space is *trivial* if it is of the form  $A \xrightarrow{s} A \times F \xrightarrow{\text{pr}_1} A$  where the section  $s$  is trivial, i.e.  $s(a) = (a, x_0)$  for some  $x_0 \in F$ .

**Theorem 13.** *Let  $M$  be a connected  $n$ -manifold without boundary. Then the fibrewise pointed space  $M \xrightarrow{\Delta} M \times M \xrightarrow{\text{pr}_1} M$  is locally trivial.*

*Proof.* We must show that  $M$  admits an open cover such that the restrictions of the above fibrewise pointed space over the elements of the cover are isomorphic to the trivial one. The proof is based on two lemmas:

**Lemma 14.** *There exists a map  $g: \mathring{B}^n \times B^n \rightarrow B^n$ , such that for every  $x \in \mathring{B}^n$  the map  $g(x, -): B^n \rightarrow B^n$  is a homeomorphism, which is identity on the boundary  $\partial B^n$  and maps the centre  $0^n$  to  $x$ .*

*Proof.* For all  $t \in \partial B^n$  let  $g(x, -)$  map the oriented segment between  $0^n$  and  $t$  linearly onto the oriented segment between  $x$  and  $t$  as shown in Figure 1. Obviously,  $g$  is continuous in both variables.  $\square$

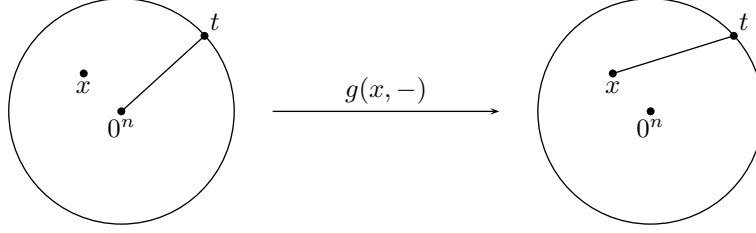


FIGURE 1. Construction of the map  $g$ .

**Lemma 15.** *For every point  $m_0 \in M$  there is an open neighbourhood  $U$  of  $m_0$  and a map  $\varphi_U: U \times M \rightarrow M$ , such that for all  $m \in U$  the map  $\varphi_U(m, -): M \rightarrow M$  is a homeomorphism and  $\varphi_U(m, m_0) = m$ .*

*Proof.* Let  $U$  be an Euclidean neighbourhood of  $m_0$  and let  $h: (B^n, 0^n) \rightarrow (\overline{U}, m_0)$  be a homeomorphism. Then we can define  $\varphi_U: U \times M \rightarrow M$  by using the map  $g$  from Lemma 14.

$$\varphi_U(m, m') := \begin{cases} h(g(h^{-1}(m), h^{-1}(m'))) & m' \in \overline{U} \\ m' & m' \in M \setminus U \end{cases}$$

$\square$

Now the claim of Theorem 13 follows easily since we may cover  $M$  by open sets  $U$  for which there exist local trivializations as in the diagram

$$(2) \quad \begin{array}{ccccc} U \times M & \xrightarrow[\approx]{(\text{pr}_1, \varphi_U)} & U \times M & \hookrightarrow & M \times M \\ (1, c_{m_0}) \left( \downarrow \text{pr}_1 \right. & & \Delta \left( \downarrow \text{pr}_1 \right. & & \Delta \left( \downarrow \text{pr}_1 \right. \\ & \xrightarrow{\quad \quad \quad} & U & \hookrightarrow & M \end{array}$$

where  $c_{m_0}$  is the constant map with value  $m_0$ .  $\square$

Since open connected manifolds are homogeneous we may strengthen Lemma 15 slightly: we can choose a point  $m_0 \in M$  and then find an open cover  $\mathcal{U}$  for  $M$  together with maps  $\varphi_U: U \times M \rightarrow M$  for all  $U \in \mathcal{U}$  such that for all  $m \in U$  the maps  $\varphi_U(m, -)$  are homeomorphisms of  $M$  and  $\varphi_U(m, m_0) = m$ . Every family  $\{(U, \varphi_U: U \times M \rightarrow M)\}$  of open sets and corresponding maps which satisfies these properties will be called a *trivialization atlas* for the fibrewise pointed space  $M \xrightarrow{\Delta} M \times M \xrightarrow{\text{pr}_1} M$ .

There are many topological constructions, like the wedge or the loop-space, the results of which explicitly depend on the choice of the base-point even when the base-point is normally dropped from the notation. With a trivialization atlas at hand we will be able to build fibrewise pointed spaces which take into account the effect of the variation of the base-point.

A functor  $F: \text{Top}_\bullet \rightarrow \text{Top}_\bullet$  is said to be *continuous* if whenever  $f: Z \rightarrow \text{Top}_\bullet((X, x), (Y, y))$  is continuous the induced map  $\bar{f}: Z \rightarrow \text{Top}_\bullet(F(X, x), F(Y, y))$ , given by  $z \mapsto F(f(z))$ , is also continuous (cf. definition of continuous functors on vector spaces in [8]). One can easily check that most of the standard functorial constructions like products, wedges, smash products, path- and loop-spaces are continuous. We will also need special notation for spaces and base-points of the values of  $F$ : if  $F(X, x) = (Y, y)$  then we write  $Y = F_{\text{sp}}(X, x)$  and  $y = F_{\text{pt}}(X, x)$ .

Let  $M$  be an open connected manifold with a trivialization atlas  $\{(U, \varphi_U)\}$ , and let  $F: \text{Top}_\bullet \rightarrow \text{Top}_\bullet$  be a continuous functor. For every  $U$  from the atlas there is a commutative diagram

$$(3) \quad \begin{array}{ccc} \bigsqcup_{m \in U} \{m\} \times F_{\text{sp}}(M, m_0) & \xrightarrow{\bigsqcup F(\varphi_U(m, -))} & \bigsqcup_{m \in U} \{m\} \times F_{\text{sp}}(M, m) \\ \downarrow F^0 \quad \downarrow \text{pr}_1 & & \downarrow \hat{F} \quad \downarrow \text{pr}_1 \\ U & \xlongequal{\quad} & U \end{array}$$

where

$$F^0: m \mapsto (m, F_{\text{pt}}(M, m_0)) \quad \text{and} \quad \hat{F}: m \mapsto (m, F_{\text{pt}}(M, m)).$$

We define a topology on the set  $\bigsqcup_{m \in U} \{m\} \times F_{\text{sp}}(M, m)$  by identifying the set  $\bigsqcup_{m \in U} \{m\} \times F_{\text{sp}}(M, m_0)$  with the topological product  $U \times F_{\text{sp}}(M, m_0)$  and requiring that the bijection

$$\bigsqcup F(\varphi_U(m, -)): U \times F_{\text{sp}}(M, m_0) \rightarrow \bigsqcup_{m \in U} \{m\} \times F_{\text{sp}}(M, m)$$

be a homeomorphism. Given another chart  $(V, \varphi_V)$  we have two possible definitions for the topology over  $U \cap V$  as seen in the diagram

$$\begin{array}{ccc} & (U \cap V) \times F_{\text{sp}}(M, m_0) & \\ & \nearrow \bigsqcup F(\varphi_V(m, -))^{-1} F(\varphi_U(m, -)) & \downarrow \bigsqcup F(\varphi_V(m, -)) \\ (U \cap V) \times F_{\text{sp}}(M, m_0) & \xrightarrow{\bigsqcup F(\varphi_U(m, -))} & \bigsqcup_{m \in U \cap V} \{m\} \times F_{\text{sp}}(M, m) \end{array}$$

However, the two topologies coincide since the map  $\bigsqcup F(\varphi_V(m, -))^{-1} F(\varphi_U(m, -))$  is a homeomorphism due to the continuity of the functor  $F$ . We conclude that the topologies on the restrictions  $\bigsqcup_{m \in U} \{m\} \times F_{\text{sp}}(M, m)$  define a unique topology on  $\bigsqcup_{m \in M} \{m\} \times F_{\text{sp}}(M, m)$ , and we denote the resulting topological space by  $M \ltimes F(M)$ .

**Proposition 16.** *Let  $M$  be an open connected manifold and let  $F$  be a continuous functor on  $\text{Top}_\bullet$ . Then  $M \ltimes F(M)$  is a fibrewise pointed space with respect to the*

projection

$$\text{pr}: \bigsqcup_{m \in M} \{m\} \times F_{\text{sp}}(M, m) \rightarrow M, \quad (m, n) \mapsto m$$

and the section

$$\widehat{F}: M \rightarrow \bigsqcup_{m \in M} \{m\} \times F_{\text{sp}}(M, m), \quad m \mapsto (m, F_{\text{pt}}(M, m)).$$

If  $A$  is a subspace of  $M$  we denote by  $A \ltimes F(M)$  the restriction of  $M \ltimes F(M)$  to  $A$ .

Let us consider a couple of examples. The identity functor on  $\text{Top}_\bullet$  assigns  $(M, m) \mapsto (M, m)$ , and the corresponding fibrewise pointed space  $M \ltimes M$  clearly coincides with our initial example  $M \xrightarrow{\Delta} M \times M \xrightarrow{\text{pr}_1} M$ . Another example is the  $n$ -fold product functor  $\Pi^n: (M, m) \mapsto (M^n, \Delta_n(m))$ , and in this case  $M \ltimes \Pi^n M$  is the fibrewise pointed space described in Section 2. The two sections are  $\widehat{\Pi}(m) = (m, \Delta_n(m))$  and  $\Pi^0(m) = (m, \Delta_n(m_0))$ .

The above construction also leads to a construction of morphisms of fibrewise pointed spaces. Let  $\nu: F \rightsquigarrow G$  be a natural transformation of continuous functors. Then the formula

$$1 \ltimes \nu: (m, x) \mapsto (m, \nu(x))$$

determines a fibre-preserving function  $1 \ltimes \nu: M \ltimes F(M) \rightarrow M \ltimes G(M)$ . This function is continuous because the following commutative diagram (where the horizontal maps are homeomorphisms by definition and  $1 \ltimes \nu$  is clearly continuous) shows that all restrictions of  $1 \ltimes \nu$  to the charts of a trivialization atlas are continuous:

$$(4) \quad \begin{array}{ccc} U \times F_{\text{sp}}(M, m_0) & \xrightarrow{\bigsqcup F(\varphi_U(m, -))} & \bigsqcup_{m \in U} \{m\} \times F_{\text{sp}}(M, m) \\ \downarrow 1 \ltimes \nu & & \downarrow 1 \ltimes \nu \\ U \times G_{\text{sp}}(M, m_0) & \xrightarrow{\bigsqcup G(\varphi_U(m, -))} & \bigsqcup_{m \in U} \{m\} \times G_{\text{sp}}(M, m) \end{array}$$

The routine proof of the following Proposition is left to the reader.

**Proposition 17.** *If  $\nu: F \rightsquigarrow G$  is a natural transformation of continuous functors then the functor  $C_\nu$  which to every pointed space  $(X, x)$  assigns the reduced mapping cone of the map  $\nu_{(X, x)}: F(X, x) \rightarrow G(X, x)$  is also continuous. Moreover, the fibrewise pointed space  $M \ltimes C_\nu(M)$  coincides with the fibrewise mapping cone of the map  $1 \ltimes \nu: M \ltimes F(M) \rightarrow M \ltimes G(M)$ .*

In particular the fibrewise pointed space  $M \ltimes \wedge^n M$  is the fibrewise mapping cone of the inclusion  $M \ltimes W^n X \hookrightarrow M \ltimes \Pi^n M$ .

We can also give an easy criterion for the fibre homotopy equivalence of fibrewise pointed constructions:

**Proposition 18.** *Let  $\nu: F \rightsquigarrow G$  be a natural transformation of continuous functors, and assume that  $\nu_{(M, m)}: F(M, m) \rightarrow G(M, m)$  is a homotopy equivalence for every  $m \in M$ . Then  $1 \ltimes \nu: M \ltimes F(M) \rightarrow M \ltimes G(M)$  is a fibre-homotopy equivalence of fibrewise pointed spaces.*

Let us conclude this section with a remark regarding more general spaces. One cannot proceed along the same lines as above:  $X$  needs not be a homogeneous

space so in general  $X \times X$  is not locally trivial as a fibrewise pointed space. This problem is even more pronounced when we consider functorial constructions like wedges where  $(X, x_0) \vee (X, x_0)$  and  $(X, x_1) \vee (X, x_1)$  need not be homeomorphic. If  $X$  is an ENR then we can view it as a deformation retract of a manifold  $M$  (namely the open regular neighbourhood of an embedding of  $X$  in some Euclidean space). Then  $X \times F(M)$  is defined and one can prove that its fiber homotopy type (as a fibrewise pointed space) depends only on the homotopy type of  $X$ . Therefore  $\mathrm{TC}(X) = \mathrm{TC}(M)$  and  $\mathrm{TC}(M)$  can be determined from the fibrewise Whitehead or Ganea definitions introduced in section 3.

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